Bifurcation Problems of Ordinary Stochastic Differential Equations

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CONTENTS

INTRODUCTION

This thesis addresses the qualitative theory of stochastic differential equations. The chosen approach is strongly influenced by ideas of Poincaré and Lyapunov, both of them investigating yet at the begin of the century solution properties of deterministic problems, which can be concluded from the form of the given equations. Poincaré created consequently a global theory, especially by the fact, that he considered from the very beginning differential equation on manifolds, whose geometry he took in account. Lyapunov developed his well-known theorems regarding the stability behaviour for solutions of differential equations, whereby the eigenvalues of the linearized problem play an outstanding role. For stochastic differential equations are eigenvalues less important. But Oseledec showed at the end of the sixties the existence of invariants for dynamical systems in measure spaces, which he was investigating at this time. These invariants were indicating some eigenvalue related properties. Utilizing this invariants – called Lyapunov exponents – it is possible to gain stability conclusions also for stochastic systems. The primary objective of this thesis was an replacement of the stability statements of the above named theory by an equivalent theorem grounded in local analytical bifurcation theory. For different reasons, explained closer in chapter 3, turns this approach out to be not an adequate one.

The investigations still gave a calculation of Lyapunov coefficients for the case of linearizations with constant coefficients.

1. FREDHOLM OPERATORS AND ANALYTICAL BIFURCATION THEORY IN BANACH SPACES

1.1 Fredholm Operators

Let F be a mapping from a Banach space X in a Banach space Y. For $x \in X$ and $y \in Y$ consider the equation:

$$Fx = y$$

Additionally, let

$$F^*x = y^*$$

the adjunct (dual) equation.

Let $N(F)$ and $R(F)$ denote null space respective range of F. Given a linear subspace M, his co-dimension is defined by

$$codim(M) \stackrel{\text{def}}{=} dim(M^{\perp})$$

where $M^{\perp}$ denotes the orthogonal complement from M.

Definition 1.1.1. $F \in L(X,Y)$ is a **linear Fredholm operator** if and only if $dim(N(F))$ and $codim(R(F))$ are finite. $ind(F) \stackrel{\text{def}}{=} dim(N(F)) - codim(R(F))$ is said to be the **index** of F.

The nonlinear operator F on $U \subseteq X$ (U open) who maps U in Y is a **Fredholm operator** if and only if:

(i) F is C^1-mapping, i.e. it exists a continuous Frechét-derivative.

(ii) $DF(x)$ is linear Fredholm operator $\forall x \in U$

If the index of $DF(x)$ is constant $\forall x \in U$ then he is called **index** of F.

Let F be a linear Fredholm operator. Then the following assertions are true:

- If $ind(F) = 0$ and $N(F) = \{0\}$ $\curvearrowright$ $Fx = y$ has an unique solution $\forall y \in Y$ and $F^{-1} \in L(Y, X)$.
- Suppose $R(F)$ is closed. For fixed $y \in Y$ is $Fx = y$ solvable if and only if $\langle x^*, y\rangle = 0 \ \forall x^* \in N(F^*)$[1)]
- The perturbed operator $F + C$ $(C \in L(X, Y))$ is likewise a Fredholm operator, if one of the following conditions is true:
  (i) C is compact
  (ii) $\|C\| < \lambda \quad , \lambda > 0$

  In this case $ind(F + C) = ind(F)$ is true.
- F^* is Fredholm operator with the subsequent properties:

$$dim(N(F^*)) = codim(R(F))$$
$$dim(N(F)) = codim(R(F^*))$$
$$ind(F) = -ind(F^*)$$

  The dual equation has a solution for fixed $y^* \in Y^*$ if and only if $\langle y^*, x\rangle = 0 \ \forall x \in N(F)$

To evaluate the Fredholm property for the operator F, the following equivalent formulation is used in many cases:

Theorem 1.1.2. *[17, vol. 2 pp. 366/367] $F \in L(X, Y)$ is Fredholm operator if and only if F is compact regularizable, i.e. $\exists R, L \in L(Y, X)$ and compact operators $S \in L(Y, Y)$, $T \in L(X, X)$ with:*

$$FR = I_Y + S$$
$$LF = I_X + T$$

1.2 Analytical Bifurcation Theory in Banach Spaces

Suppose X and Y are two real or complex Banach spaces and F is a mapping from $\mathbb{K} \times X$ in Y. We consider the following equation:

$$F(\varepsilon, x) = 0 \qquad x \in X \quad \varepsilon \in \mathbb{K} \tag{1.2.1}$$

$\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

1) $\langle x^*, y\rangle$ denotes the value of the linear continuous functional x^* at y.

Definition 1.2.1. (ε_0, x_0) is called **bifurcation point** if and only if:

a) $F(\varepsilon_0, x_0) = 0$

b) There exist two series of solutions for (1.2.1) $\{(\varepsilon_n, x_n)\}$, $\{(\varepsilon_n, y_n)\}$ converging towards (ε_0, x_0) with $x_n \neq y_n$ for all n.

Corollary. *If* (ε_0, x_0) *is bifurcation point for* (1.2.1) *and* F C^1*-mapping on a neighborhood of* (ε_0, x_0) *then* $F_x^{-1}(\varepsilon_0, x_0)$ *is not element of* $L(Y, X)$.

This results immediately from the uniqueness statement in the implicit function theorem, which is violated by b).

Theorem 1.2.2. *Let*

(i) $F:\ U(0,0) \subseteq \mathbb{K} \otimes X \mapsto Y,\ C^k$*-mapping* $\quad k \geq 1,\ F(0,0) = 0$

(ii) $F_x(0,0)$ *Fredholm operator with index* κ, *dim*$(N(F_x(0,0))) = n$

Then is the solution of $F(\varepsilon, x) = 0$ *equivalent to the solution of a* $(n - \kappa)$*-dimensional equation system over* $\mathbb{K}$ *in n+1 variables. These equations are also called* bifurcation equations.[2)]

Proof. (outline)[3)] Using the implicit function theorem, we transform (1.2.1) in the bifurcation equations. For this, we split F_x in two parts, one of them a bijective mapping (thus allowing solvability). The decomposition is conducted by appropriate projection operators. The second part – which cannot treated with the implicit function theorem – is finite dimensional because the Fredholm property.
Let $B \stackrel{\text{def}}{=} F_x(0,0)$. Then exist two projection operators P, Q with:

$$\begin{aligned} P:\ X \mapsto X, \qquad & P(X) = N(B) \\ Q:\ Y \mapsto Y, \quad & (I - Q)(Y) = R(B) \end{aligned}$$

They can be represented in the following form[4)]:

$$Px = \sum_{i=1}^{n} \langle y_i^*, x \rangle x_i$$
$$Qy = \sum_{i=1}^{m} \langle x_i^*, y \rangle y_i$$

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2) The exact form of the bifurcation equations will be derived in the proof.
3) See [17, vol. 2 pp. 279/280] for the complete proof.
4) The essential reason for the existence of the projections lies in the finite dimensionality of $N(B)$ and $R(B)^{\perp}$. This allows the representation of X and Y as direct topological sum [17, p. 369].

The y_i^* form in combination with the x_i a biorthogonal base. Similarly, the x_i^* and y_i constitute such a base. (1.2.1) is equivalent to:

$$(I-Q)F(\varepsilon, y+z) = 0 \qquad \text{(a)}$$

$$QF(\varepsilon, y+z) = 0 \qquad \text{(b)}$$

$$y = (I-P)x \quad z = Px$$

(a) is solvable in y and we get y = y(ε,z). Substituting y in (b) by this solution gives the bifurcation equations:

$$QF(\varepsilon, y(\varepsilon, z) + z) = 0 \qquad \text{(c)}$$

The dimension of (c) is codim(I-Q) = n$-\kappa$. The number of variables is evidently n+1. On the other hand, solving (c) gives z = z(ε) and x = y(ε,z(ε)) + z(ε) solves (1.2.1). □

Remark. The described reduction of the original system to the bifurcation equations is called *Lyapunov-Schmidt procedure.*

2. CENTER MANIFOLDS

2.1 Deterministic Case

Before proceeding with the stochastic case, we introduce some concepts for deterministic center manifolds.
We consider a system of ordinary autonomous differential equations in R^n in the neighborhood of a critical point x_0:

$$\dot{x} = f(x)\,, \quad f(x_0) = 0 \tag{2.1.1}$$

respectively the decomposition in a linear and a nonlinear part:

$$\dot{x} = A(x) + g(x) \tag{2.1.2}$$

A is the matrix of the first derivatives in the point x_0. It is evident, that f has been assumed as C^1 vector field ($r \geq 1$). Without loss of generality choose $x_0 = 0$. If this is not true per se, we perform an appropriate coordinate shift. If g still contains linear parts, we add them to A, i.e. we require for g and his Jacobi matrix to vanish in 0:

$$g(0) = Dg(0) = 0$$

Now, we transform A in Jordan normal form, thus splitting (2.1.2) in three equation systems:

$$\dot{x_1} = A_s x_1 + g_s(x_1, x_2, x_3) \tag{2.1.3}$$

$$\dot{x_2} = A_u x_2 + g_u(x_1, x_2, x_3) \tag{2.1.4}$$

$$\dot{x_3} = A_c x_3 + g_c(x_1, x_2, x_3) \tag{2.1.5}$$

The following table shows the relation between the spectrum of A and the single A_i:

Matrix	Real part of eigenvalues
A_s	< 0
A_u	> 0
A_c	$= 0$

The subscript in A_i characterizes the stability behaviour of the associated solutions. For equations with linearization A_s and A_u exist the classical results from LYAPUNOV, connecting the stability respective instability of the solution with the sign of the eigenvalues.[15, p. 312 ff.]. A more generalized result is given by the theorem from HARTMAN-GROBMAN for hyperbolic fixed points of a dynamical system. It states, that the stability of a (non-linear) dynamic system in the neighborhood of such a fixed point is exclusively defined by his linearization [2, p. 287]. More precisely, it exist a (local) homeomorphism between the flow, generated from the original system, and the flow of the linear system[1)]. This mapping preserves also the direction of time. The fixed point is said to be *hyperbolic*, if all of the eigenvalues of the matrix for the linearized system are different from zero and not purely imaginary. In a neighborhood of the point the existence of two manifolds W_s, W_u can be ensured. The manifolds points converge to the fixed point, if they are mapped by the flow, which is generated from the differential equation. They do so for W_u, if $t \to -\infty$ (*local unstable manifold*) and for W_s, if $t \to \infty$ (*local stable manifold*). In a way, both manifolds are non-linear analogies of the (generalized) eigen spaces E_s and E_u, associated to the matrices A_s and A_u. W_s is tangent to E_s and analogously W_u is tangent to E_u [8, p. 14].
Up to now, nothing has been said about the matrix A_c. Whereas W_u and W_u allow a definition immediately from stability criteria, this is not appropriate for the definition of a center manifold below. Those elements are able to show stable or unstable behavior.

Definition. Let $\varphi(t,x) \stackrel{\text{def}}{=} (\varphi_s(t,x), \varphi_u(t,x), \varphi_c(t,x))$ denote a solution of (2.1.3-5) with initial value x.
A **local $\mathbf{C^k}$ center manifold** for (2.1.3-5) is a set $M \subset R^n$

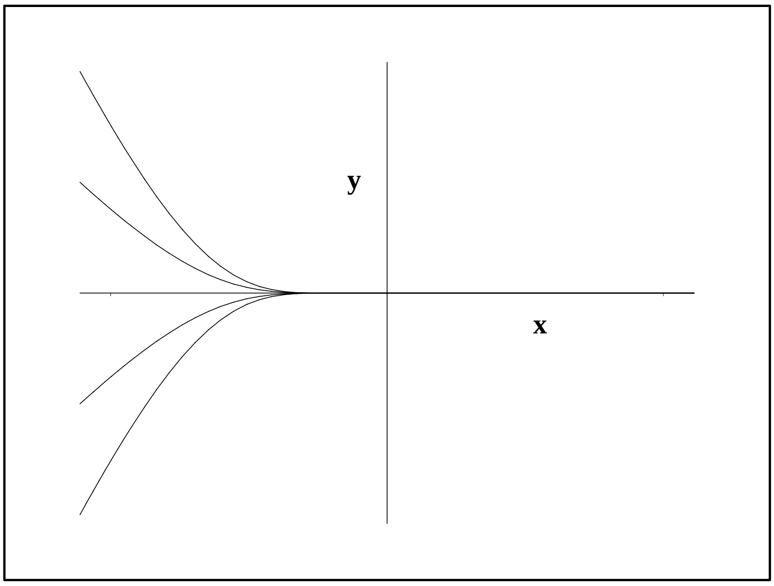
$$M = \{(x_c, h_s(x_c), h_u(x_c)) \mid \|x_c\| \le \varepsilon,\ x_c \in E_c,\ \varepsilon > 0\}$$

with
$h_s : E_c \mapsto E_s,\ h_u : E_c \mapsto E_u$ are C^k-mappings, $k \ge 1$
$h_{s,u}(0) = Dh_{s,u}(0) = 0$

Additionally, the set shall have the following invariance property:

$\varphi(t, x_c, h(x_c)) \in M$ and $\forall (x_c, h(x_c)) \in M : \|\varphi_c(t, x_c, h(x_c))\| \le \varepsilon$
$h(x_c) \stackrel{\text{def}}{=} (h_s(x_c), h_u(x_c))$

1) Assigning additional requirements to the linearizations eigenvalues (also called *resonance conditions*) it is possible to achieve even diffeomorphic resp. holomorphic equivalence. This is expressed in the theorems of STERNBERG, POINCARÉ and SIEGEL (see [3]).



Example. *[16, p. 194 following A.Kelly]*
Consider the system

$$\dot{x} = x^2$$
$$\dot{y} = -y$$

with solution

$$x = -\frac{1}{t + C_1}, \ \ y = C_2 e^{-t} \tag{*}$$

The C_i depend on the initial conditions. The associated Matrix for the linearized (in 0) equation has two eigenvalues 0 and -1. The x-axis equals to the space E_c, the y-axis coincides with E_s. The following relation defines a center manifold W_c in the phase space:

$$y(x) = \begin{cases} Ce^{1/x} & : \ x < 0 \\ 0 & : \ x \geq 0 \end{cases}$$

with $C \stackrel{\text{def}}{=} C_2 e^{C_1}$.
Elimination of t in (*) gives the term for $x < 0$. The construction guarantees the invariance of W_c.

One can realize clearly the relatively complex structure for the center manifold. She divides into infinite many submanifolds – determined through the variation of the initial value.

Under the same assumptions as to stable (unstable) manifolds exist theorems, assuring the existence of center manifolds for equations of type (2.1.3-5)[3, pp. 56/57 without proof]. Contrary to $W_{s,u}$ – which possesses the same differentiability properties – the former is generally only C^{r-1}-manifold. The lacking differentiability extends also to the case $r = \infty$ and even to analytical vector fields [3, ib][2)]. Also is a center manifold not at all unique (cf. [14, p. 445 ff.] for the estimation of the distance between two different center manifolds). These disadvantages are partly canceled by the following theorem:

Theorem (Reduction Principle). *[3, pp. 56/57 without proof]*
Suppose the right side of (2.1.3-5) is C^2-mapping. Then is (2.1.3-5) in a neighborhood of the singularity topologically equivalent to (2.1.5) and the standard saddle*:*

$$\dot{x} = -x \qquad x \in E_s$$
$$\dot{y} = y \qquad y \in E_u$$

Remarks.

- The theorem reduces essential properties of (2.1.3-5) to the system (2.1.5), which has often a much lower dimension compared with the original system. Consider e.g. the bifurcation properties of parameter dependent equations. Then, the eigenvalues also depend on these parameters. For a significant number of problems we have the situation of only a few eigenvalues having vanishing real parts for fixed parameter values. Thus, the problem is equivalent to a system of equations with a dimension equal to the dimension of the associated generalized eigen spaces for this eigen values.
- The reduction principle can be proven under much weaker assumptions (esp. in this local version). See [14] for additional information.

2.2 Stochastic Center Manifolds

2.2.1 Basic Concepts

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We consider a measurable flow $\{\vartheta_t\}$ of measure preserving transformations over Ω (cf. appendix about the definition). Let I be the index range for t, with $i \subseteq R^1$. $\{\vartheta_t\}$ is supposed to be ergodic (see appendix).

2) The example above confirms the assertion. W_c is C^∞-manifold but not analytic.

Furthermore let $\varphi : I \times \Omega \times R^n \mapsto R^n \quad (t, \omega, x) \mapsto \varphi(t, \omega, x)$ a mapping, satisfying the properties:

(C1) $\varphi(t, \cdot, x)$ is $\mathcal{F}, \mathcal{B}^n$ -measurable for all $t \in I$ and $x \in R^n$
($\mathcal{B}^n$ is the σ-field of Borel sets in R^n)

It exist a ϑ_t-invariant set Ω_0 with measure 1 in Ω, so that for all $\omega \in \Omega_0$ the following conditions are true:

(C2) $\varphi(t, \omega, x)$ is continuous in $(t, x) \in I \times R$

(C3) It exist a $k \geq 1$, so that $\varphi(t, \omega, \cdot)$ is $C^{k,\alpha}$-diffeomorphism [3] for all $t \in I$

(C4) *(cocycle property)*
$\varphi(t + s, \omega, \cdot) = \varphi(t, \vartheta_s \omega, \cdot) \circ \varphi(s, \omega, \cdot) \quad \text{for all } t, s \in I$

Definition 2.2.1. The introduced mapping φ is called **cocycle** of $C^{k,\alpha}$-diffeomorphisms over ϑ_t on R^n.[4]

Remarks.

- Let $\dot{x} = f(x)$ a system of ordinary autonomous differential equations in R^n. Assume a solution $\varphi(t)$ for the system with initial condition $\varphi(0) = \xi$ has been given. Fixing ξ and introducing

$$y(t) \stackrel{\text{def}}{=} \left.\frac{\partial \varphi(t)}{\partial \xi}\right|_{\xi = x_0},$$

we obtain $y(t)$ as a solution for the variation system:

$$\dot{y} = \frac{\partial f}{\partial x}(\varphi_{x_0}(t))\, y \qquad (*)$$

For variable ξ is φ a flow T_t over $R^n$and (*) is an element of a family of differential equations

$$\left\{ \dot{y} = \frac{\partial f}{\partial x}(T_t \xi)\, y \qquad y(0) = y_0 \right\}_\xi$$

[3] A bijective mapping between manifolds (here $R^n \mapsto R^n$), which along with their inverse satisfies continuous differentiability up and including order k. The derivative of order k is Hölder-continuous with Hölder coefficient α.

[4] The cocycle is often seen as mapping from R^n to R^n (for (t, ω)) fixed).

The family's fundamental matrices

$$A(t,\xi) \quad : \quad y(0) \mapsto y(t)$$

constitute a deterministic cocycle

$$A(t+s,\xi) = A(t,T_s\xi) \circ A(s,\xi) \quad \text{for all } s^{5)} \tag{**}$$

justified from the fact, that (**) is an equivalence to

$$y(t+s) = \tilde{y}(t)$$

y and $\tilde{y}$ are defined by two initial value problems

$$\begin{aligned} \dot{y}(t) &= \frac{\partial f}{\partial x}(T_t\xi)\, y(t) \qquad & y(0) &= y_0 \\ \dot{\tilde{y}}(t) &= \frac{\partial f}{\partial x}(T_t T_s\xi)\, \tilde{y}(t) \qquad & \tilde{y}(0) &= y(s) \end{aligned}$$

If we consider the first equation for the argument t+s and subtract the equation for $\tilde{y}$, then follows from the group property for T_t for every fixed s

$$g(t) = \frac{\partial f}{\partial x}(T_{t+s}\xi)\, g(t) \qquad\qquad g(0) = 0$$

with

$$g(t) \stackrel{\text{def}}{=} y(t+s) - \tilde{y}(t)$$

$$\curvearrowright \quad g(t) = ce^{\int \dots dt} \quad \overset{g(0)=0}{\curvearrowright} \quad g(t) \equiv 0$$

This finishes the proof.

For a more general dynamical system on a manifold, the differentials build a cocycle if the tangential bundle is assumed to be a direct product[6)] and for the associated matrices we have the relation

$$(dT^{t+s})_x = (dT^t)_{T^s x}(dT^s)_x$$

The subscript denotes the base of the corresponding fiber [11, p. 199].

- With view of the kind of concatenation for the expressions in (C4) the cocycles are called *multiplicative cocycles*. If we have a matrix formulation and we switch to the logarithm for the matrix norm, we get an *additive* cocycle (" $\circ$ " is replaced by $+$).

[5)] The solution for the original system exist in general of course only locally. Hence, all of the considered t,s,t+s have to be assumed as elements of a maximal existence interval.

[6)] A counterexample is the Möbius band.

Definition 2.2.2. A probability measure μ on $\Omega \times R^n$ is called **invariant** relative to the cocycle φ, if

(i) the restriction of μ to Ω is P.

(ii) μ is invariant relative to the *skew-product flow* Θ_t:

$$\begin{aligned} \Theta_t \ : \ \Omega \times R^n &\mapsto \Omega \times R^n \\ (\omega, x) &\mapsto (\vartheta_t \omega, \varphi(t, \omega, x)) \end{aligned}$$

i.e. $\mu\Theta_t = \mu$ (q.v. appendix)

Theorem 2.2.3. *μ is invariant related to φ if and only if $\Theta_t(\omega, x)$ forms a strict-sense stationary process on $(\Omega \times R^n, \mu)$ with values in $(\Omega \times R^n, \mathcal{F} \otimes \mathcal{B}^n)$*

Proof. See [6, p. 24]

2.2.2 The Ergodic Theorem of Oseledec

We will now consider the derivative of φ in $x \in R^n$, in other words, a mapping between tangent spaces of R^n:
Let $v \in T_x R^n$ and $\sigma(\tau)$ a curve in R^n with $\sigma(0) = x$ and $\frac{d\sigma}{d\tau}\big|_{\tau=0} = v(x)$. The derivative of φ in x for fixed (t, ω) is then defined by

$$\begin{aligned} T\varphi(t, \omega, x) : \quad & T_x R^n \mapsto T_{\varphi(t,\omega,x)} R^n \\ T\varphi(t, \omega, x) v \stackrel{\text{def}}{=} & \frac{d}{d\tau}\bigg|_{\tau=0} \varphi(t, \omega, \sigma(\tau)) \end{aligned}$$

It is well known, that this definition actually not depends on σ, but only on v [5, p. 45]. In the considered case leads the introducing of coordinates to a representation by the Jacobi matrix.
Also the mapping $T\varphi(t, \omega, x)$ is a cocycle in respect of Θ_t on $\Omega \times R^n$. Interpreting T_φ as mapping of the whole tangent bundle, it represents a so called *measurable isomorphism* [11, pp. 210/211]. I.e. in our case a mapping, which preserves the tangent bundle and whose restriction to mappings between the single fibers is a linear isomorphism. Measurable isomorphisms build the general model of mappings used by Oseledec in his original paper (the definition and some properties can be found in a very tense form in [11, p. 212 ff.]).
Nowadays, the various original statements will be mostly collected in a single theorem. Such a version shall now be given. It has been adapted to the intended use in stochastics, but in company with the original paper [11, pp. 217-230] the proof is comprehensible.

Definition 2.2.4. The **Lyapunov coefficient** of the linearized cocycle $T\varphi(t,\omega,x)$ in the point $x \in R^n$ in direction $v \in T_x R^n$, $v \neq 0$ under the influence of ω is the number:

$$\lambda(\omega,x,v) \stackrel{\text{def}}{=} \limsup_{t\to\infty} \frac{1}{t} \ln \|T\varphi(t,\omega,x)v\|$$

Theorem 2.2.5 (Multiplicative Ergodic Theorem of Oseledec). *Consider a given cocycle φ on R^n over ϑ_t and an invariant (see Definition 2.2.2) ergodic (see appendix) probability measure on $\Omega \times R^n$. If the condition*

$$\int_{\Omega\times R} \ln^+ \sup_{0\le t\le 1} \|(T\varphi(t,\omega,x))^{\pm 1}\| \, d\mu < \infty$$

$$\ln^+(x) \stackrel{\text{def}}{=} \max(0, \ln(x))$$

is true, then exists a Θ_t-invariant subset $\Gamma \subset \Omega \times R^n$ with measure 1 and real numbers $\lambda_r < \ldots < \lambda_1$ $(1 \le r \le n)$ with multiplicities n_i : $\sum_{i=1}^r n_i = n$, so that for all $(\omega,x) \in \Gamma$ is true:

(i) It exists a measurable invariant decomposition

$$R^n = E_r(\omega,x) \oplus \ldots \oplus E_1(\omega,x), \quad \dim E_i(\omega,x) = n_i$$
$$E_i(\Theta_t(\omega,x)) = T\varphi(t,\omega,x)E_i(\omega,x) \quad i \in \{1\ldots r\}$$

(ii) The Lyapunov exponent of the linearized cocycle satisfies

$$\lambda(\omega,x,v) = \lambda_i \quad \Leftrightarrow \quad v \in E_i(\omega,x)$$

The tuples (λ_i, n_i) compose the *Lyapunov spectrum* belonging to μ. The $E_i(\omega,x)$ are called *Oseledec spaces.*

Remarks.

- The E_i can be interpreted as mapping $\Omega \times R^n \mapsto G_{n_i}(R^n)$. $G_{n_i}(R^n)$ denotes the Grassmann bundle built up from the n_i-dimensional subspaces of the tangent spaces of R^n (More information is found in [1, p. 145 ff. and p. 175 ff.]. The manifolds open sets generate a σ-field of Borel sets over $G_{n_i}(R^n)$. The term "measurable" in (i) relates to this σ-field. Invariance is equivalent to the second relation in (i).
- The integrability condition ensures the existence and certain invariances for the limes in the definition of Lyapunov coefficients by using the ergodic theorem from Birkhoff [11, pp. 207-209]. She is automatically satisfied, if R^n is replaced by a compact manifold. However, this is too restrictive for the stochastic case, since it requires a compact probability space.

2.2.3 Stochastic Differential Equations and Cocycles

This chapter investigates the existence of the formerly introduced cocycles for dynamical systems, which are representable as stochastic differential equations. For this purpose these equations must be defined on the entire R^1. This will become more clear, if we calculate the inverse of a cocycle. Since the base flow ϑ_t forms a group we get $\varphi(0,\omega,\cdot) = \varphi^2(0,\omega,\cdot)$, i.e. $\varphi(0,\omega,\cdot)$ is the identity. Utilizing the cocycle property we get

$$\varphi^{-1}(t,\omega,\cdot) = \varphi(-t,\vartheta_t\omega,\cdot)$$

and negative times on the right hand side come in view. Thus, we have to find in our case a probability space, which allows the definition of a Wiener process for all times. The following construction has been given in [6, p. 33 ff.]. Starting point is the canonical model of a Wiener space:

$$\begin{array}{ll} \Omega \overset{\text{def}}{=} \left\{\omega \in C(R^+,R^m)\,|\omega(0)=0\right\}, & m \geq 1 \\ \mathcal{F} & \text{(completed) Borelian } \sigma\text{-field of events} \\ P & \text{Wiener measure on } (\Omega,\mathcal{F}) \end{array}$$

Over this space a m-dimensional Wiener process is defined, which can be identified with $\omega(t)$ [12, p. 17 ff. and p. 30 ff.]. Considering two copies of this space, we can define the Brownian motions

$$\begin{aligned} \omega^+(t) &\overset{\text{def}}{=} \omega_1(t) \qquad && t \geq 0 \\ \omega^-(t) &\overset{\text{def}}{=} \omega_2(-t) \qquad && t \leq 0 \end{aligned}$$

(The subcripts denote the respective space)

The product of the both to ω^+ and ω^- associated spaces forms the requested probability space:[7)]

$$(\Omega,\mathcal{F},P) \overset{\text{def}}{=} (\Omega^+\times\Omega^-, \mathcal{F}^+\otimes\mathcal{F}^-, P^+\times P^-)$$

The Wiener process in $(\Omega,\mathcal{F},P)$ is defined by

$$\omega(t) \overset{\text{def}}{=} \begin{cases} (\omega^+(t),0) & : \ t \geq 0 \\ (0,\omega^-(t)) & : \ t < 0 \end{cases}$$

For the proof that $\omega(t)$ is a Wiener process, it is sufficient to show the independence of the increments. The other properties follow immediately from

[7)] To simplify things, we will use the notations from the canonical model above again.

the construction. The increments Gauss distribution shortens this even further to show non-correlation [6, p. 34].

It exist a flow of "shifts" $\{\vartheta_t\}$ on Ω:

$$\vartheta_t\omega(s) \stackrel{\text{def}}{=} \omega(s+t) - \omega(t)$$

Lemma 2.2.6. *P is invariant and ergodic relative to the flow $\{\vartheta_t\}$*

PROOF. See [6, p. 35 ff.]

Check also [12, pp. 32-37] for various invariance properties of the Wiener measure.

After this preparations, we can formulate the system of Stratonović differential equations

$$dx_t = X_0(x_t)\,dt + \sum_{i=1}^{m} X_i(x_t) \circ dW_i(t) \qquad x_0 = x \in R^n \tag{2.2.1}$$

The W_i are independent Wiener processes. The systems domain is the entire time axis.

Theorem 2.2.7. *Let the vector fields X_i fulfill the following conditions:*

- *X_0 is global $C^{k,\alpha}$-function* $\quad k \geq 1, \quad \alpha > 0$
- *X_i is $C^{k+1,\alpha}$-function* $\quad k \geq 1, \quad \alpha > 0, \quad i \in \{1 \dots m\}$
- *The derivatives (including the one for the highest order) for all X_i's are bounded*

Then defines (2.2.1) for almost all ω a local cocycle of $C^{k,\beta}$-diffeomorphisms over ϑ_t with $\beta < \alpha$.

(No proof)

Prior to applying the multiplicative ergodic theorem we give some propositions regarding the linearizations of (2.2.1).

Theorem 2.2.8. *The linearized cocycle $Tx(t,\omega,x) : R^n \mapsto R^n, \; v \mapsto v_t$ is generated by the following Stratonović equation:*

$$dv_t = A_0(x_t(\omega,x))v_t\,dt + \sum_{i=1}^{m} A_i(x_t(\omega,x))v_t \circ dW_i(t) \qquad v_0 = v \in R^n \tag{2.2.2}$$

Here, $x_t(\omega, x)$ is solution for (2.2.1). The $A_i(y)$ are the Jacobi matrices of X_i in y.

(No proof)

Remark. Arnold gives no proofs in his paper. They can be concluded with some effort from a much more generalized theorem in [10, pp. 160-180].

2.2.4 Definition of Stochastic Center Manifolds

In the deterministic problem we have been considered a fixed point with neighborhood, which allows a linearization. In the general case we have exclusively cocycle, linearized cocycle and an ergodic, invariant measure. But one would be mistaken yet in the deterministic case by the term fixed point – defined through the vanishing vector field of the equation – if we had not an alternative description. He also is literally a fixed point of the associated flow – a point, not moving under the influence of the flow in time. Following this path a fixed point is then a probabilistic element in R^n, which remains stationary under the the impact of the cocycle[8].
Let us go back to the more general premises of Oseledec's theorem. We apply a coordinate transformation to the cocycle, which enables without loss of generality the fixed point to be in 0:

$$\hat{\varphi}(t, \omega, x, v) \stackrel{\text{def}}{=} \varphi(t, \omega, x + v) - \varphi(t, \omega, x) \qquad x \in R^n,\ v \in T_x R^n$$

It is simple to show, that $\hat{\varphi}$ forms a cocycle with fixed point in the origin [6, p. 48 Theorem 4.1].
To satisfy the assumption for the ergodic theorem, we have to find for $\hat{\varphi}$ a probability space and an ergodic, invariant measure on the product space of this space with R^n[9]. Aside from this, we have to fulfill the integrability condition for the linearized cocycle. The probability space is

$$(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}) \stackrel{\text{def}}{=} (\Omega \times R^n, \mathcal{F} \otimes \mathcal{B}^n, \mu^{10)})$$

The construction 'embeds' the copy of R^n containing the base point of the tangent space into the new robability space. The final form for the measure on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$ is defined by

$$\hat{\mu} \stackrel{\text{def}}{=} \widehat{P} \otimes \delta_0 \qquad (\delta_0 \ldots \text{Dirac measure in } 0).$$

[8] From my point of view it is even for stochastic differential equations difficult, to describe a fixed point through vanishing drift and diffusion terms (the right side of the equation), albeit this can be still useful in other contexts .

[9] The tangent space in x is identified with R^n.

[10] μ is the required measure from the assumptions to the ergodic theorem.

$\hat{\mu}$ is ergodic, since μ is ergodic and δ_0 generates always sets of measure 0 or 1. Thus also the invariant sets have these measures. The invariance results from a construction of an adequate skew-product flow [6, p. 50]

$$\begin{aligned} \hat{\Theta}_t : \quad & \hat{\Omega} \times R^n \mapsto \hat{\Omega} \times R^n \\ (\hat{\omega}, 0) & \mapsto (\Theta_t \hat{\omega}, \hat{\varphi}(t, \hat{\omega}, 0)) \end{aligned}$$

The linearization in the new origin coincides with those of the original cocycle, i.e. the integrability condition is satisfied. Since the ergodic theorem remains valid and the Lyapunov exponents are the same for both linearized cocycle, we can further on assume a cocycle with fixed point zero and switch back to our old notation. The theorem from Oseledec provides a partition of R^n in a direct sum of Oseledec spaces:

$$R^n = \bigoplus_{i=1}^{r} E_i(\omega, 0)$$

The invariance of E_i is implicated by the fixed point property:

$$T\varphi(t, \omega, 0) E_i(\omega, 0) = E_i(\Theta_t(\omega, 0)) = E_i(\vartheta_t \omega, 0)$$

From this, it follows also

$$R^n = \bigoplus_{i=1}^{r} E_i(\vartheta_t \omega, 0) \qquad \forall t \in R$$

From now we skip the last argument, because the precondition, that the linearization point corresponds to 0.

Collecting the Lyapunov exponents with the same sign in different groups, we get

$$\begin{aligned} E_s(\omega) &\stackrel{\text{def}}{=} \bigoplus_{\lambda_i < 0} E_i(\omega) \\ E_u(\omega) &\stackrel{\text{def}}{=} \bigoplus_{\lambda_i > 0} E_i(\omega) \\ E_c(\omega) &\stackrel{\text{def}}{=} \bigoplus_{\lambda_i = 0} E_i(\omega) \end{aligned}$$

Using the Ansatz

$$\psi(t, \omega, x) \stackrel{\text{def}}{=} \varphi(t, \omega, x) - T\varphi(t, \omega) x$$

it follows

$$\varphi(t,\omega,x) = T\varphi(t,\omega)x + \psi(t,\omega,x)$$

Projecting the cocycle φ on the Oseledec spaces E_s, E_u, E_c we get the final partition

$$\varphi_s(t,\omega,x_s,x_u,x_c) = T\varphi_s(t,\omega)x_s + \psi(t,\omega,x_s,x_u,x_c) \tag{2.2.3}$$
$$\varphi_u(t,\omega,x_s,x_u,x_c) = T\varphi_u(t,\omega)x_u + \psi(t,\omega,x_s,x_u,x_c) \tag{2.2.4}$$
$$\varphi_c(t,\omega,x_s,x_u,x_c) = T\varphi_c(t,\omega)x_c + \psi(t,\omega,x_s,x_u,x_c) \tag{2.2.5}$$

We can now define a stochastic center manifold for this system. To this end, we consider a set of mappings of the following form:

$$X \overset{\text{def}}{=} \{h : E \mapsto F \,|\, h(\omega,\cdot)\ :\ E_c(\omega) \mapsto E_s(\omega) \oplus E_u(\omega)\,,\ \text{(i) is true}\}$$
$$E \overset{\text{def}}{=} \bigcup_{\omega\in\Omega} \{\omega\} \times E_c(\omega)$$
$$F \overset{\text{def}}{=} \bigcup_{\omega\in\Omega} \{\omega\} \times E_s(\omega) \oplus E_u(\omega)$$

(i) h is almost surely continuous and bounded relative to the following for almost all ω defined norm [6, pp. 58/59]:

$$\|h(\omega,\cdot)\|_{\infty,\omega} \overset{\text{def}}{=} \sup_{x\in E_c(\omega)} (\|h_s(\omega,x)\|^s_\omega + \|h_u(\omega,x)\|^u_\omega)$$

$h_{s,u}(\omega)$ denotes the projections from h to the respective component $E_{s,u}(\omega)$ and furthermore

$$\|x\|^s_\omega \overset{\text{def}}{=} \int_0^\infty e^{-(\lambda_s+2\beta)\tau}\, \|T\varphi_s(\tau,\omega)x\|\, d\tau \qquad x \in E_s(\omega)$$
$$\|x\|^u_\omega \overset{\text{def}}{=} \int_0^\infty e^{(\lambda_u-2\beta)\tau}\, \|T\varphi_u(-\tau,\omega)x\|\, d\tau \qquad x \in E_u(\omega)$$
$$\|x\|^c_\omega \overset{\text{def}}{=} \int_0^\infty e^{-2\beta\tau}\, \|T\varphi_c(\tau,\omega)x\|\, d\tau \qquad x \in E_c(\omega)$$

λ_s is the largest negative Lyapunov coefficient and λ_u the smallest of the positive coefficients. The norms exist for all β with:

$$\lambda_s + 4\beta < 0, \qquad \lambda_u - 4\beta > 0, \qquad \beta > 0$$

It is possible to define a pseudo-metric d on X as an expectation value by the norm $\|\cdot\|_{\infty,\omega}$:

$$d(h,\hat{h}) \stackrel{\text{def}}{=} d(h_s,\hat{h}_s) + d(h_u,\hat{h}_u)$$
$$d(h_{s,u},\hat{h}_{s,u}) \stackrel{\text{def}}{=} \int_\Omega \frac{\|h_{s,u}(\omega,\cdot) - \hat{h}_{s,u}(\omega,\cdot)\|_{\infty,\omega}}{1 + \|h_{s,u}(\omega,\cdot) - \hat{h}_{s,u}(\omega,\cdot)\|_{\infty,\omega}}\, dP$$

The transition to equivalence classes of a.s. equality $(X\backslash\hat{})$ strengthens the condition for the vanishing of the function d to a degree, that d becomes a metric[11].The associated convergence is convergence in probability.

Lemma 2.2.9. *(X,d) is a complete metric space.*

PROOF. See [6, p. 62 ff.]

Remark. The norm $\|\cdot\|_\omega^{s,u,c}$ has another remarkable property. She is equivalent to the ordinary in $E_{s,u,c}(\omega)$ induced euclidian norm, i.e there exist random variables $a_{s,u,c}(\omega)$ and $b_{s,u,c}(\omega)$ with

$$a_{s,u,c}(\omega)\,\|\cdot\|_{R^n} \le \|\cdot\|_\omega^{s,u,c} \le b_{s,u,c}(\omega)\,\|\cdot\|_{R^n}$$

[6, p. 57 Lemma 4.2.][12]

Let us now consider a subset from (X,d), depending on a constant $L > 0$ with:

(i) $D^i h(\omega,\cdot)$ exists for all $j \in \{0\dots k\}$

(ii)

$$h(\omega,\cdot) = 0$$
$$D^1 h(\omega,\cdot) = 0 \quad \text{if } k \ge 1$$

(iii) $\|D^j h_{s,u}(\omega,\cdot)\|_{\infty,\omega} \le \frac{L}{2} \qquad j \in \{0\dots k\}$

(iv) $\|D^k h_{s,u}(\omega,x) - D^k h_{s,u}(\omega,\hat{x})\|_{\infty,\omega} \le \frac{L}{2}\|x-\hat{x}\|_\omega^c \qquad \forall x,\hat{x} \in E_c(\omega)$

The subset of all h having the properties (i)–(iv) over a set with measure 1 is denoted by $A_k(L)$. The set can depend on h.

[11] From now, let $X \stackrel{\text{def}}{=} X\backslash\hat{}$.

[12] The property is useful in allowing us below to transfer probability from the norm function to the leading coefficients. This gives raise to special cases of Lipschitz conditions, which have applications in the theory of stochastic differential equations. For more information look for the term *'functional Lipschitz'* in [13, p. 195].

Definition 2.2.10. Let $h \in A_k(L)$. A set

$$M(\omega) \overset{\text{def}}{=} \{(x, h(\omega, x)) | \, x \in E_c(\omega)\}$$

is called **local stochastic $\mathbf{C^{k,1}}$-center manifold** for the cocycle φ, if she conforms to the following condition:
It exist a random neighborhood $U(\omega)$ of the origin in R^n with:

$$\varphi(t, \omega, y) \in U(\vartheta_t \omega) \longrightarrow \varphi(t, \omega, y) \in M(\vartheta_t \omega)$$
$$\forall y \in M(\omega) \cap U(\omega) \quad \forall t \in I$$

Following a theorem from Boxler *(ib p. 102 theorem 6.1)* such a $C^{k,1}$-center manifold exist, if the k*th* derivative of the cocycle is Lipschitz-continuous (see definition 2.2.1 property (C3)).
Coming back to the Stratonovic differential equations we see, that these produce according theorem 2.2.7 only a Hölder-continous $C^{k,\beta}$-cocycle. But a continous derivative provides us (mean value theorem) at least with a Lipschitz constant. Thus, we have in any case a Lipschitz continous $C^{k-1,1}$ cocycle. It follows, that the stochastic differential equation possesses a $C^{k-1,1}$ center manifold.

Theorem 2.2.11 (Reduction Principle).

(i) Consider the mapping

$$y_c \mapsto \varphi_c(t, \omega, y_c, h(\omega, y_c))$$

Then is true:

0 is (asymptotic) fixed point of φ if and only if 0 is (asymptotic) fixed point of the reduced system above.

(ii) Let $x = (x_s, x_u, x_c)$ sufficient small. Then exists an $y \in E_c(\omega)$ as well as a positive measurable random variable $k(\omega)$ and an $\eta > 0$ so that the following becomes true:

$$\|\varphi_c(t, \omega, x_s, x_u, x_c) - \varphi_c(t, \omega, y_c, h(\omega, y_c))\| \overset{a.s}{\leq} k(\omega) e^{-\eta t}$$
$$\|\varphi_s(t, \omega, x_s, x_u, x_c) - h^s_{\vartheta_t \omega}(\varphi_c(t, \omega, y_c, h(\omega, y_c)))\| \overset{a.s}{\leq} k(\omega) e^{-\eta t}$$
$$\|\varphi_s(-t, \omega, x_s, x_u, x_c) - h^s_{\vartheta_{-t} \omega}(\varphi_c(-t, \omega, y_c, h(\omega, y_c)))\| \overset{a.s}{\leq} k(\omega) e^{-\eta t}$$

for all $t \geq 0$

Remark. Theorem 2.2.11 compares with [6, p. 120, theorem 8.1]. There you can find the complete proof. The difference in the formulation of both theorems originates from the usage of the above mentioned equivalent norm.

3. ANALYTICAL BIFURCATION THEORY AND CENTER MANIFOLDS

This chapter gives justifications, why it is not adequate to apply deterministic bifurcation theory to stochastic dynamical systems. For this, we begin with some motivational similarities between those and deterministic center manifolds. Starting from this, the plan was an extension to systems of stochastic ordinary differential equations. But this approach is not justified, how some theoretical considerations and – most notably – counter examples show.

We consider a system of ordinary differential equations:

$$\dot{x} = Ax + f(x, \lambda), \quad f(0,0) = f_x(0,0) = 0, \quad \lambda \in R^{1)} \tag{3.1}$$

Here, f is a mapping from $R^{n+1} \times R$ to R^{n+1} and C^1-mapping relative to x. A is a constant matrix of the form

$$A = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

B is supposed to have no eigenvalues with vanishing real part. Hence, (3.1) is a special case of the system considered at the beginning of chapter 2. Like there we can write the decomposition

$$\dot{y} = g(y, z, \lambda) \tag{3.2a}$$

$$\dot{z} = Bz + k(y, z, \lambda) \tag{3.2b}$$

Using the Lyapunov-Schmidt procedure for the system (3.2a,3.2b) means to solve (3.2b) for z (This is possible because B and k-s properties in a sufficient small neighborhood) and the subsequent insertion of the solution in the second part of (3.2a), which has been set to 0. It follows the bifurcation equation

$$G(x, \lambda) \overset{\text{def}}{=} g(y, z(y, \lambda), \lambda) = 0$$

On the other hand is it sufficient for stability assertions to consider (3.2a) around a fixed point of (3.1). This is a conclusion from the reduction principle. However, in this case the variable $z(y, \lambda)$ has to be replaced by the

[1)] λ could also be an element of a Banach space.

expression $h = (h_s, h_u)$ from the definition of the center manifold and we consider:

$$\hat{G}(y, \lambda) \stackrel{\text{def}}{=} g(y, h(y, \lambda), \lambda) = \dot{y}$$

With this assumptions, following theorem is true:

Theorem (Hale). *The generated flows from the equations*

$$\dot{y} = G(y, \lambda)$$
$$\dot{y} = \hat{G}(y, \lambda)$$

are homeomorph. The homeomorphism preserves direction of time.

It is known (from construction) that G and $\hat{G}$ have the same zeros in a neighborhood of (0,0). Thus the proof can be reduced essentially to show, that the signs of G and $\hat{G}$ between the two zeroes are the same.

Let us lead over to stochastic equations. For linearizations of stochastic differential equations, having diffusion matrices with deterministic noise term σ as factor, it is possible to give an asymptotic series expansion for the Lyapunov coefficients ([4]):

$$\lambda(\sigma) = \lambda(0) + c_1\sigma + \cdots + \mathcal{O}(\sigma^n)$$

Here, $\lambda(0)$ is an eigenvalue of the unperturbed problems. These kind of expansions are in general true only for small perturbations σ. That means that an additive partition of Lyapunov exponents in eigenvalue and perturbed part is valid only under certain circumstances.
The subsequent example results from P. Boxler (ib pp. 139-142) and shows a similar difference as in the example above, this time for center manifolds.

Consider the Stratonovic system

$$du_t = a\, u_t\, dt + \sigma_1 u_t \circ dW_1(t)$$
$$dv_t = (b\, v_t + c\, u_t^2)\, dt + \sigma_2 v_t \circ dW_2(t) \qquad \sigma_1, \sigma_2 > 0$$

The Lyapunov coefficients shall be 0 and 1. This can be simply accomplished, because the linearized system is decoupled in 0 and the solutions can explicitly stated (It follows then $a = 0$, $b < 0$). Satisfying an approximation theorem for stochastic center manifolds [6, p. 136] which also covers the deterministic case, the center manifold h(ω,y) has the representation

$$h(\omega, y) = \left(\sum_{n=0}^{\infty} e^{bn} e^{-\sigma_2 W_2(-n)(\omega)} \times \ldots \right.$$

$$\left. \ldots \times \int_0^1 e^{b(1-s)} e^{\sigma_2 W_2(1-s)(\vartheta_{-n-1}\omega)} e^{2\sigma_1 W_1(s-n-1)(\omega)} \, ds \right) y^2 + \mathcal{O}(\|y\|^3) \qquad (*)$$

$$y \in E_c(\omega)$$

$\mathcal{O}$ denotes a specific random variant of the Landau symbol, with an exact definition which is not important in this context.
Calculating the expectation value, we get

$$Eh(\cdot, \omega) = -\frac{c}{2\sigma_1^2 + b + \frac{\sigma_2^2}{2}} \, y^2 + \mathcal{O}(\|y\|^3) \quad ^{2)}$$

On the other side it is seen easily, that the approximation for the manifold of the unperturbed problem ($\sigma_1 = \sigma_2 = 0$ in (*)) results in

$$h(\cdot) = -\frac{c}{b} \, y^2 + \mathcal{O}(\|y\|^3)$$

what is different compared with the calculated expectation value.

Both examples gave rise to the decision, not to follow the way sketched initially in this chapter. This was made even more difficult by the fact, that the vector fields on right sides of the considered differential equations in the stochastic case are not representable by differentiation of the cocycles [3)].

[2)] Now an ordinary Landau symbol.

[3)] This means the differentiation in time, which is in the most cases not achievable for stochastic processes. On the other hand are deterministic flows *defined* by this differentiation.

4. CALCULATION OF LYAPUNOV EXPONENTS FOR LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Consider the following system of Itô equations:

$$dx_t = Ux_t\,dt + \sum_{k=1}^{m} V^k x_t \circ dW_t^k$$

The $U = \{u_{ij}\}$ and $V^k = \{v_{ij}\}$ $k \in \{1 \ldots m\}$ are constant $(n \times n)$-matrices. The W^k form m independent one-dimensional Wiener processes. x_t is a n-dimensional stochastic process and additionally a solution of the differential equation system with $x_0 = x \neq 0$. Since the Lyapunov exponents have the form $\lim_{t\to\infty} \frac{1}{t} \ln \|x_t\|$, we consider the process $y_t = \ln \|x_t\|$ and apply the Itô formula to them. It follows

$$dx_t^i\,dx_t^j = \sum_{k=1}^{m} \sum_{l,p=1}^{n} v_{i,l}^k v_{j,p}^k\, x^l x^p\ dt$$

and

$$\frac{\partial y_t}{\partial x^i} = \frac{x^i}{\|x_t\|^2}, \qquad \frac{\partial^2 y_t}{\partial x^i \partial x^j} = \frac{\delta ij}{\|x_t\|} - \frac{2x^i x^j}{\|x_t\|^2}$$

and y_t satisfies the equation

$$dy_t = \sum_{i=1}^{n} \frac{x_t^i}{\|x_t\|^2} \left(\sum_{j=1}^{n} u_{ij} x_t^i\,dt + \sum_{k=1}^{m} \sum_{j=1}^{n} v_{ij}^k x_t^j \circ dW_t^k \right) + \cdots$$

$$\cdots + \frac{1}{2} \sum_{i,j}^{n} \left(\frac{\delta_{ij}}{\|x_t\|^2} - \frac{2x_t^i x_t^j}{\|x_t\|^4} \right) \sum_{k}^{m} \sum_{l,p}^{n} v_{il}^k v_{jp}^k\, x_t^l x_t^p\ dt$$ [1)]

With $z_t \stackrel{\text{def}}{=} \frac{x_t}{\|x_t\|}$ we get the following differential equation over the unit sphere:

$$y_t = y_0 + \int_0^t z_t^T U z_t - \|z_t\|^{-2} z_t^T \hat{V}\hat{V}^T z_t + \frac{1}{2}\ \text{trace}\ (\hat{V}\hat{V}^T)\,dt + \cdots$$

$$\cdots + \sum_{k=1}^{m} \int_0^t z_t^T V^k z_t \circ dW_t^k$$

[1)] δ_{ij} denotes the Kronecker symbol.

$\tilde{V}$ is a matrix with the vector $V^k z_t$ as kth column. For $t \to \infty$ we get

$$\lim_{t\to\infty} \frac{1}{t} y_t = \lim_{t\to\infty} \frac{1}{t} \left(\int_0^t z_t^T U z_t - \|z_t\|^{-2} z_t^T \hat{V} \hat{V}^T z_t + \frac{1}{2} \text{ trace } (\hat{V}\hat{V}^T)\, dt \right) + \cdots$$
$$\cdots + \sum_{k=1}^{m} \lim_{t\to\infty} \frac{1}{t} \int_0^t z_t^T V^k z_t \circ dW_t^k$$

The last summand vanishes almost surely, since the integrands are bounded. The other limit exist a.s. if we impose a certain kind of ellipticity condition [7, p. 130] to $\tilde{V}\tilde{V}^T$.

Altogether follows

$$\lim_{t\to\infty} \frac{1}{t} \ln \|x_t\| = \lambda \text{ a.s.}$$

or

$$\lim_{t\to\infty} \|x_t\| = 0 \text{ a.s. for } \lambda < 0 \qquad \text{resp.}$$
$$\lim_{t\to\infty} \|x_t\| = \infty \text{ a.s. for } \lambda > 0$$

The last conditions show the λ's relation towards stability conclusions.

ERGODIC THEORY

This section provides some auxiliary concepts from ergodic theory.

Let $(\Omega, \mathcal{B})$ be a measurable space and T a measurable transformation on Ω.

Definition E1. A measure μ on a measurable space $(\Omega, \mathcal{B})$ is called **invariant**, if:

$$\mu(A) = \mu(T^{-1}(A)) \qquad \forall A \in \mathcal{B}$$

where $T^{-1}(A)$ denotes the inverse image for A.

Conversely, we can assume a measure space $(\Omega, \mathcal{B}, \mu)$ and give the following definition. In view of the application as probability measure, in this chapter μ is assumed to be a finite and normed measure.

Definition E2. A bijective mapping $T : \Omega \mapsto \Omega$ is called **measure preserving**, if T and T^{-1} are measurable and satisfy the condition:

$$\mu(T(A)) = \mu(A) \qquad \forall A \in \mathcal{B}$$

Definition E3. A group of measure preserving mappings $\{T_t\}$ $T \in \mathbf{R}$ or $T \in \mathbf{C}$ is called a **flow** on Ω.

The flow property can also be expressed as an operator property in the Hilbert space of quadratic integrable (complex) functions on Ω. I.e., is P a point in Ω and P_t his image under T_t, than follows from the bijectivity for all T_t

$$\int f(P)\, d\mu = \int f(P_t)\, d\mu$$

for all functions f in L_1 (and L_2). Denoting the linear operator on L_2, mapping $f(P)$ to $f(P_t)$ for fixed t with U_t, than follows with the usual scalar product in L_2:

$$\langle U_t f, U_t g \rangle = \langle f, g \rangle \qquad \forall f, g \in L_2$$

I.e. U_t is unitary for all t and the U_t's build a group. From now, all considered flows are assumed to be measurable and to satisfy the following definition:

Definition E4. The flow $\{T_t\}$ is called **measurable** in t, if for all $A \in \mathcal{B}$ the set $\{(P,t)\}$ $P_t \subset A$ is measurable in $\Omega \times \{t\}$.

The next theorem follows from the property, that the U_t build a group of unitarian operators:

Theorem E5. *Let $\{U_t\}$ a group of unitarian operators on a Hilbert space $\mathfrak{H}$. Then the following assertions are true:*

Every f in $\mathfrak{H}$ has a mean value (with respect to strong convergence):

$$\lim_{T-S\to\infty} \frac{1}{T-S} \int_S^T U_t f \, dt = f^*$$

f^* is invariant for all U_t, i.e. $U_t f^* = f^*$

Proof. See [9, p. 24 ff.]

Additionally for all invariant φ is true:

$$\langle f, \varphi \rangle = \langle f^*, \varphi \rangle \qquad (*)$$

Thereby, f^* is unique determined.

(*) is due essentially to the fact that norm convergence implicates weak convergence and the unitarity of U_t. The uniqueness is not trivial. [9] uses certain properties of the partition of the unit associated with the spectral representation of U_t.
The invariancy of f^* can also be considered as solution of an eigenvalue problem with eigenvalue 1.
If this eigenvalue is simple: $f^* = c\varphi$, then c is calculable from (*) with the Ansatz $\varphi \overset{\text{def}}{=} \varphi_0$ and f^* is representable in the form:

$$f^* = \frac{(f, \varphi_0)}{(\varphi_0, \varphi_0)} \varphi_0 \qquad (**)$$

Corollary (Statistical Ergodic Theorem). *Let $\mathfrak{H} = L_2(\Omega)$. Than is true:*
Every function $f(P) \in L_2$ possesses a norm convergent time mean $f^*(P)$ in $L_2(\Omega)$:

$$\lim_{T-S\to\infty} \left\| \frac{1}{T-S} \int_S^T f(P_t) \, dt - f^*(P) \right\| = 0 \qquad (***)$$

f^* is invariant under the flow and by $\langle f, \varphi \rangle = \langle f^*, \varphi \rangle$, which is true for all invariant φ, unique.

Definition E6. The flow is called **ergodic**, if for all $f \in L_2$ the time mean in Ω is a constant almost everywhere.

In the ergodic case the eigenvalue in (**) is simple, since all invariant functions are equal to their time means. Therefore (ergodicity!) they are constant and consequentially multiples of each other. I.e., (**) with $\varphi_0 = 1$ and $\mu(\Omega) = 1$ implies:

$$f^* = \frac{(f,1)}{(1,1)} = \int f\, d\mu$$

This formula corresponds to the usual "time mean = space mean" description of ergodicity.
In this case, the measure μ is also called *ergodic* (compare the different approachs leading to definition E1 resp. E2).

Definition E7. The flow is called **metrically transitive** if every measurable, invariant $A \subset \Omega$ has measure 0 or 1.

Remark. Invariancy is considered as invariancy of the characteristic function $\chi_A(P)$ for A.

Theorem E8. *Ergodicity is equivalent to metrical transitivity.*

Remark. It should be reminded that we assumed μ as finite. Without this prerequisite some of the assertions remain true. However, the proof requires considerably different auxiliary concepts.

Proof. (following [9, p. 30 ff.]) A general conclusion from metrical transitivity is the fact, that constants are the only measurable, invariant functions (up to sets of measure 0). This is due to the fact, that the set of points in Ω giving the function in question the same fixed value is invariant. Let A such a set, associated with an invariant f. Then follows $f(P) = f(P_t)\ \forall P \in A$. This means, also P_t belongs to A, in other words, A is invariant. But A has measure 0 or 1, hence f is essentially constant. But the time mean is an invariant function, i.e. metrical transitivity implies ergodicity.
If we assume conversely ergodicity, than follows that all invariant functions are constant. The characteristic function for invariant sets is *per definitionem* invariant, hence almost everywhere constant. This implicates metrical transitivity. □

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